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Air-supported structures: the state of the art

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Fig. 1.

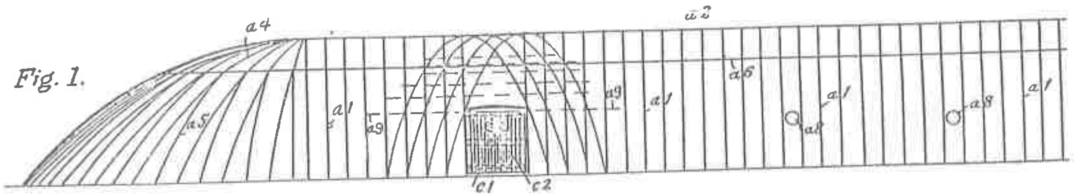
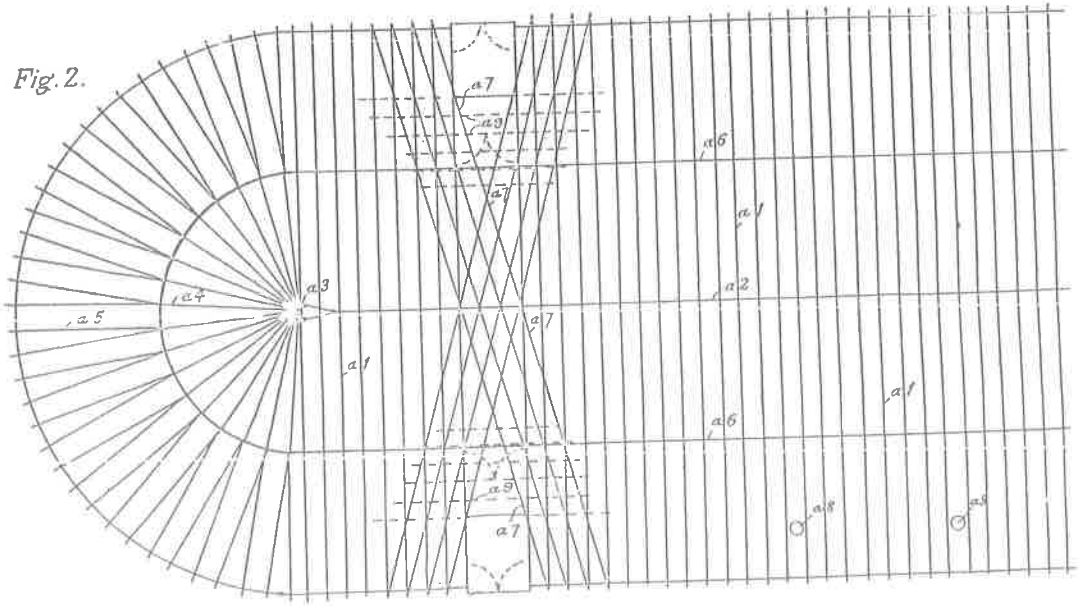


Fig. 2.



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FORM FINDING AND CUTTING PATTERNS FOR AIR SUPPORTED STRUCTURES

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SYNOPSIS:

The paper discusses the problems of form finding and production of cutting patterns for air supported structures. A method is proposed for simultaneous form and cutting pattern finding which produces principal stresses under inflation pressure in the directions of the warup and weft of the fabric.

Appendices are included giving:

A: An introduction to those topics of differential geometry most relevant to air supported structures.

B: The equilibrium equations of a membrane subject to internal pressure with principal stresses following the directions of geodesic coordinates on the surface.

INTRODUCTION

Form finding can be stated to be the definition of the geometry of the surface of an air supported structure under internal pressure but no other loads (except perhaps own weight). The behaviour of a form under external applied loads is of course of fundamental importance in deciding what is an acceptable form but the methods of form finding and the methods of analysis under load differ considerably and are therefore worthy of separate consideration.

The production of cutting patterns is a mainly geometrical operation to find the shapes of material which when joined together form the surface of the structure. The production of cutting patterns can be a separate operation after the form finding or can be an integral part of the form finding as shown later in this paper.

A proper understanding of form finding and the production of cutting patterns requires some knowledge of differential geometry - the branch of geometry concerned with the study of surfaces. Appendix A is an introduction to those topics of differential geometry most relevant to air supported structures.

2. REQUIREMENTS OF FORM

The fundamental requirement is that under internal pressure all principal stresses in the membrane are tensile. A necessary but not sufficient condition for this is that at least one of the two principal radii of curvature point inwards. This is less stringent than the corresponding condition for prestressed membranes with no internal pressure (tents) which is that one principal radius of curvature must point inwards and one outwards. Fortunately the problem that this condition is not sufficient is removed if form finding methods which specify tensions initially and then calculate shape are used rather than methods working in the opposite direction.
The stress distribution under internal pressure should be reasonably uniform and can be thought of as being controlled on two levels of scale. Firstly the overall state of stress is controlled by the overall geometry of the surface. Secondly this overall state of stress is modified locally by, for example, cutting pattern inaccuracies or the incompatibility of strains in the surface and boundary supports.

For certain airhouses a uniformly stressed (soap film) surface is suitable. However this does put certain limitations on form. For example if the boundary of the airhouse has corners then the stress in the surface in the direction of maximum curvature must tend to zero at the corners. If this is not done the surface will be in the same plane as the boundary at the corner.

The equilibrium equations of an air supported surface show that there can not be discontinuities in curvature without discontinuities of stress and therefore strain. Discontinuities in strain are not possible unless introduced intentionally or otherwise during manufacture. Therefore in practice the structure will deform to remove the discontinuity in curvature. Thus a cylindrical airhouse with spherical ends will deform locally at the sphere/cylinder junction to remove the discontinuity of curvature.

3. CUTTING PATTERN REQUIREMENTS

If a net with equal distances between nodes is laid on a surface then the angle α between the threads of the net will change continuously (see fig. 1). This problem was first studied by Tschebycheff and the angle α is given by

$$K = - \frac{1}{\sin\alpha} \frac{\partial^2 \alpha}{\partial u \partial v}$$

where K is the Gaussian curvature and u and v are distances along the threads (see appendix A for proof of this result). The threads of the net can be identified with the warp and weft directions of a coated fabric and therefore it can be seen that the angle between the warp and weft directions must continuously change. To limit the maximum distortion of the fabric the size of the panels must be limited. In practice this is not usually a problem since except for small structures made with large fabric width, it is the fabric width which limits the panel size.

Fabric comes in rolls and therefore the panels should be as fig. 2a rather than 2b to avoid wastage. This can be done by choosing the seam lines to be geodesic lines on the surface. If Δ is the distance between two adjacent geodesics (see fig. 3) and v the distance along the geodesics then (see appendix A)

$$K = - \frac{1}{\Delta} \frac{\partial^2 \Delta}{\partial v^2}$$

again where K is the Gaussian curvature. Thus the shape of the panel is determined solely by the Gaussian curvature. If K is positive the panel will be "convex" both sides, if K is negative both sides will be "concave". If K is zero the surface is developable and the sides can be parallel.

The question as to how accurate a cutting pattern has to be is difficult to answer quantitatively. As was the case with stress distribution the problem exists at two levels of scale. Overall illogicalities in cutting pattern may still produce an acceptable airhouse (but of a different shape to that envisaged). On the other hand small errors especially near boundaries can produce unacceptable results. Certainly it would seem that the stiffer the fabric the more accurate the cutting pattern has to be. Air supported structures are more tolerant of cutting pattern inaccuracies than are tents due partly to the less stringent requirements on curvature mentioned in section 2. The cutting pattern should take into account strains due to inflation but it seems that boundary strains are normally ignored and this causes no real problems.

In general there will not be a state of uniform stress in the surface. This means that the coating will have to take shear stresses unless the principal stress directions coincide with the warp and weft directions of the fabric. It is possible to organise the form and cutting pattern such that the principal stresses are parallel to the warp and weft directions under inflation pressure only. A method for doing this is described in section 4.3.

4. METHODS

4.1. Form Finding

The equilibrium equations of the surface under inflation pressure give the relationship between stresses in the surface and the geometry of the surface. The equilibrium equations consist of three simultaneous partial differential equations and there are therefore three unknowns. The unknowns can either be the stresses or the geometry. Thus two approaches are possible.

i) Define geometry, inflation pressure and boundary forces and then calculate stresses. There are three unknown stresses (two direct stresses and one shear stress) and therefore the stresses can be calculated directly - the membrane is statically determinate. The defined geometry must be that which applies under inflation pressure rather than under no load. The no load geometry can then be found by consideration of the strains necessary to produce the inflation stresses. The methods by which the equilibrium equations can be solved for stresses are similar to those used for considering applied loads and therefore will not be considered further in this paper.

ii) Define stresses, inflation pressure and boundary geometry and then calculate the resulting surface geometry. Again there are effectively three unknowns (for example the x, y, z coordinates of the surface). If the stresses are defined to be uniform then this process is physically equivalent to a soap film determining its shape on a given boundary with a given inflation pressure. A method for using this approach to form finding using both uniform and non uniform stress distributions is described in section 4.3.

4.2. Cutting Patterns

The major task in producing cutting patterns is to find geodesic lines across the surface. It should be noted that there is a geodesic in every direction through every point on a surface and that a geodesic is uniquely determined by a starting point and starting direction. Alternatively a geodesic can be determined by two points (see fig. 4). In the case of analytically defined surfaces the geodesics can be found by solution of the differential equation for geodesics (see appendix A). Except for simple cases the equation will have to be solved numerically. If the surface is defined by numerical data then it is usual to consider the surface as faceted. If this is the case a geodesic will cross the edge between two facets such that the angle between the geodesic and the edge is the same both sides (see fig. 5). This can be seen to be the case by considering a tape laid over the surface. If the facets are large and curved then change in direction of a geodesic across a facet may need to be considered.

Once the geodesics have been found it is a purely geometrical problem to flatten the panels formed by each pair of geodesics.

4.3. Simultaneous form and cutting pattern finding

This method was developed to fulfill the following requirements:

1. To simultaneously find a form in equilibrium and geodesic lines on the surface.

2. To have a stress distribution in the surface under inflation pressure such that the principal stress directions coincide with the warp and weft directions of the fabric. The simplest stress distribution is the uniform (or soap film) but many others are possible.

Fortunately these requirements are compatible and yield particularly simple equilibrium equations (see appendix B). A computer is necessary for the process since a relaxation method is used involving a large number of repeated calculations. Firstly the boundary and required stress distribution are defined together with some initial starting geometry (usually flat). Then the nodes on the surface are moved one by one in three different directions (see fig. 6 the movements refer to node B):

1. Normal to the surface according to equilibrium requirements.

2. In the plane of the surface perpendicular to the geodesics to make (angle 1 + angle 2 + angle 3) = (angle 4 + angle 5 + angle 6). Note that angle 1 + angle 2 + angle 3 + angle 4 + angle 5 + angle 6 will in general not equal 360° due to the curvature of the surface.

3. In the plane of the surface parallel to the geodesics to make length AB equal length BC.

Movement of each node upsets the conditions at each adjacent node and therefore each node has to be moved a large number of times until the errors are acceptably small.

The fact that (angle 1 + angle 2 + angle 3) = (angle 4 + angle 5 + angle 6) ensures geodesic lines on the surface. It is not necessary to make lengths along the geodesics equal, however this does prevent bunching of the nodes along the geodesics.

For the more complex types of stress distribution it is necessary to recalculate the stresses in the surface as the surface moves. Fortunately this is relatively easy due to the simplicity of the equilibrium equations and can be done directly with no iteration or solution of simultaneous equations.

Once a satisfactory shape has been found then the pattern of nodes on the surface makes production of cutting patterns particularly simple. The surface consists of strips between the geodesics and each strip consists of flat triangles with folds between (see fig. 7).

Angles 1, 2, 3 on fig. 7 are known as are lengths such as AB and BC. Thus the shape of each side of the strip when flattened out can be calculated directly. It only remains to calculate the relative position of the ends of the sides and the shape of the strip is completely determined.

In many instances it is necessary to have a constant maximum strip width determined by the fabric width. This can be achieved by automatically moving the ends of the geodesics along the boundary during the form finding process.

The form finding process is economical in terms of computer storage and calculation time and requires considerably less time than an analysis under applied load. This is because only the geometry is unknown whereas under applied load both the stresses and geometry are unknown. The production of cutting patterns is especially fast because it involves direct calculations only.

Figs. 8, 9, 10 show drawings of form and cutting pattern for an air supported structure on a rectangular plan. A rectangular plan was chosen for the example since a plan with corners is more difficult to handle than one with smooth boundaries. This is due to the rapid stress change necessary to give sufficient curvature at the corners. An extra geodesic is introduced at the corners to help model the stress change, but these geodesics do not represent an actual seam line.

The maximum panel width is not constant in fig. 10 but this can easily be achieved by moving the ends of the geodesics or seam lines along the boundary automatically during the analysis. This has been done for a number of other cases including tents. Figs. 11, 12 show cutting patterns for the same form with different constant panel widths. These cutting patterns were obtained from fig. 10 by graphical methods on a drawing board.

APPENDIX A. Introduction to differential geometry

This appendix is included because of the insights afforded by differential geometry into the problems of form finding and production of cutting patterns. An attempt has been made to introduce those areas of differential geometry most relevant to air supported structures in as brief a way as possible. The vector notation has been adopted rather than the more modern tensor notation since it is probably easier for engineers and architects to assimilate. Differential geometry is often used as a vehicle for the introduction of the tensor notation so that the notation can then be used for topics where it must be employed such as relativity.

One of the best books on the subject is Lectures on Classical Differential Geometry by Struik (1) and the notation adopted here is similar but not identical to Struik's.

A surface in three dimensional space can be represented as $f(x,y,z) = 0$. For example

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 - z = 0 \text{ represents a hyperbolic paraboloid.}$$

However for differential geometry it is convenient to introduce two parameters, u and v such that

$$x = X(u,v)$$

$$y = Y(u,v)$$

$$z = Z(u,v)$$

The same surface can be represented in an infinite number of different parametric forms. Thus two of the ways in which the hyperbolic paraboloid can be expressed are:

$$x = au \cosh v$$

$$y = bu \sinh v$$

$$z = u^2$$

and

$$x = a(u + v)$$

$$y = b(u - v)$$

$$z = 4uv$$

If v is kept constant at a particular value and u varied then x, y and z will change their values and a <u>line</u> traced out in space. If the same thing is done for a number of different values of v then a number of lines will be traced out. Similarly u can be kept constant and v varied to produce a second set of lines crossing the first set (see fig A1).

Thus the lines u = constant and v = constant form a system of curvilinear coordinates on the surface. The lines will not in general cross at right angles nor will the distance between intersections be constant. Differential geometry studies the properties of the surface by consideration of the behaviour of these lines on the surface.

The x, y, z coordinates of a point on the surface can be considered as the components of a vector $\bar{r}$ from the origin to the point on the surface. Note a — over a letter is used to signify a vector quantity. The vector from point A to point B on Fig. A2 is equal to:

$$\bar{r}\,(u+\delta u, v) - \bar{r}\,(u, v)$$

$$= \frac{\partial \bar{r}}{\partial u}\,\delta u \quad \text{as} \quad u \to 0$$

Thus $\frac{\partial \bar{r}}{\partial u}$ is a vector tangent to the surface in the direction of the u coordinate line. Similarly $\frac{\partial \bar{r}}{\partial v}$ is a vector tangent to the surface in the direction of the v coordinate.

$\frac{\partial \bar{r}}{\partial u}$ is normally written $\bar{r}_u$ and similarly

$$\frac{\partial \bar{r}}{\partial v} = \bar{r}_v$$

$$\frac{\partial^2 \bar{r}}{\partial u^2} = \bar{r}_{uu}, \quad \frac{\partial^2 \bar{r}}{\partial u \partial v} = \bar{r}_{uv} \quad \text{and so on.}$$

First Fundamental Form

In fig. A2 the vector AD is equal to $\bar{r}_u \delta u + \bar{r}_v \delta v$

Therefore the length AD is given by

$$AD = \delta s = \sqrt{[(\bar{r}_u \delta u + \bar{r}_v \delta v).(\bar{r}_u \delta u + \bar{r}_v \delta v)]}$$

where the . signifies the scalar product. Thus

$$s^2 = E\delta u^2 + 2F\delta u\,\delta v + G\,\delta v^2 \qquad \text{——— } 1$$

where $E = r_u . r_u$

$F = r_u . r_v$

$G = r_v . r_v$

This formula is of great importance in differential geometry and the quantities E, F and G recur constantly. The equation (1) is termed the first fundamental form.

Second fundamental form

The first fundamental form is concerned with lengths on a surface and the second fundamental with curvature of the surface.

Fig. A3 shows a line LM across the surface. S is the arc length along the line, $\bar{N}$ is a unit vector normal to the surface, $\bar{t}$ is a unit vector tangential to the line and $\bar{p}$ is a unit vector perpendicular to both $\bar{N}$ and $\bar{t}$.

$$\bar{t} = \bar{r}_u \frac{du}{ds} + \bar{r}_v \frac{dv}{ds} \qquad \text{——— 2a}$$

$$\bar{N} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|} \qquad \text{——— 2b}$$

$$\bar{P} = \bar{N} \times \bar{t} \qquad \text{——— 2c}$$

where x denotes the vector product and $|\bar{r}_u \times \bar{r}_v|$ the magnitude of $\bar{r}_u \times \bar{r}_v$.

Since $\bar{p}$ is perpendicular to $\bar{N}$, it lies in the plane of $\bar{r}_u$ and $\bar{r}_v$. Thus $\bar{p}$ can also be expressed as

$$\bar{p} = \frac{-\bar{r}_u\left(G\frac{dv}{ds} + F\frac{du}{ds}\right) + \bar{r}_v\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)}{\sqrt{(EG-F^2)}} \qquad \text{——— 2d}$$

It can be seen that this formula fulfills the criteria $\bar{p}.\bar{t} = 0$ and $\bar{p}.\bar{p} = 1$.

As S varies the trihedron formed by the three vectors $\bar{t}$, $\bar{p}$, $\bar{N}$ rotates according to the curvature of the surface and the line LM drawn upon it.

The normal curvature is defined as

$$K_n = \frac{d\bar{t}}{ds}.\bar{N} = -\bar{t}.\frac{d\bar{N}}{ds} \quad (\text{since } \bar{t}.\bar{N} = 0) \qquad \text{— 3a}$$

the geodesic curvature as

$$K_g = \frac{d\bar{t}}{ds}.\bar{p} = -\bar{t}.\frac{d\bar{p}}{ds} \qquad \text{——— 3b}$$

and the twist of the surface as

$$K_t = \frac{d\bar{p}}{ds}.\bar{N} = -\bar{p}.\frac{d\bar{N}}{ds} \qquad \text{——— 3c}$$

It should be noted that a unit vector can only rotate and not change its length. Therefore differentiating a unit vector with respect to S effectively gives the change in direction of the vector per unit length along the line LM. For example $\bar{t}.\bar{t} = 1$ and therefore $\bar{t}.\frac{d\bar{t}}{ds} = 0$ and $\frac{d\bar{t}}{ds}$ is perpendicular to $\bar{t}$.

K_n and K_t can be calculated by differentiating equations 2 and scalar multiplying by $\bar{N}$. Hence since $\bar{r}_u.\bar{N} = \bar{r}_v.\bar{N} = 0$

$$K_n = \frac{e\,du^2 + 2f\,du\,dv + g\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \qquad \text{3d}$$

$$K_t = \frac{(Ef-eF)\,du^2 + (gE - eG)\,du\,dv + (Fg - fG)\,dv^2}{\sqrt{(EG - F^2)}\,(E\,du^2 + 2F\,du\,dv + G\,dv^2)} \qquad \text{3e}$$

$$\left.\begin{aligned} \text{where } e &= \bar{r}_{uu}.\bar{N} = -\bar{r}_u.\bar{N}_u \\ f &= \bar{r}_{uv}.\bar{N} = -\bar{r}_u.\bar{N}_v = -\bar{r}_v.\bar{N}_u \\ g &= \bar{r}_{vv}.\bar{N} = -\bar{r}_v.\bar{N}_v \end{aligned}\right\} \qquad \text{3f}$$

The expression $e\,du^2 + 2f\,du\,dv + g\,dv^2$ is termed the <u>second fundamental form</u>. Thus

$$K_n = \frac{\text{Second fundamental form}}{\text{First fundamental form}}$$

Lines of Curvature

It can be seen that K_n and K_t depend only on the ratio $\frac{dv}{du}$ and hence only on the direction of the line on the surface. It can be shown that there exist two values of $\frac{dv}{du}$ at each point on the surface at which K_n is maximum and minimum (unless K_n is constant in all directions) and that at these values of $\frac{dv}{du}$ $K_t = 0$. The directions on the surface given by these two values of $\frac{dv}{du}$ are perpendicular to each other and the corresponding values of K_n are called the principal curvatures.

Lines on the surface in the directions of the principal curvatures are termed lines of curvature. The lines of curvature can be found by integrating equation 3e for $K_t = 0$. If the parametric lines on the surface are chosen as the lines of curvature then $F = \bar{r}_u.\bar{r}_v = 0$ since the lines are orthogonal. Thus in order that K_t can be zero equation 3e shows that $f = 0$.

Geodesics

K_n and K_t were expressed in terms of E, F, G, e, f, g and $\frac{dv}{du}$. In a similar way equation 3b for the geodesic curvature, K_g, can be expressed as a function of E, F and G, the derivatives of E, F and G with respect to u and v, $\frac{dv}{du}$ and $\frac{d^2v}{du^2}$. Thus the geodesic curvature of a line depends on $\frac{d^2v}{du^2}$ whereas K_n and K_t did not.

A line on a surface with zero geodesic curvature is termed a geodesic. The relationship between v and u along a geodesic can be

found by solving the differential equation $K_g = 0$. The form of this equation shows that there is a geodesic in every direction through every point on a surface and that a geodesic is uniquely determined by a starting point and a starting direction.

Gaussian Curvature and Gauss' Theorem

Gaussian curvature is defined to be the product of the two principal normal curvatures. It can be shown from equation 3d that

$$\text{Gaussian curvature} = K = \frac{eg-f^2}{EG-F^2} \qquad 4$$

$$\bar{N}, \quad \bar{a} = \frac{\sqrt{G}\bar{r}_u + \sqrt{E}\,\bar{r}_v}{\sqrt{(2EG + 2\sqrt{E}\sqrt{G}F)}}, \quad \bar{b} = \frac{\sqrt{G}\bar{r}_u - \sqrt{E}\bar{r}_v}{\sqrt{(2EG - 2\sqrt{E}\sqrt{G}F)}}$$

form a set of three mutually perpendicular unit vectors. Therefore using the properties of the scaler product $eg - f^2$ can be written

$$\begin{aligned} eg - f^2 &= (\bar{r}_{uu}.\bar{N})(\bar{r}_{vv}.\bar{N}) - (\bar{r}_{uv}.\bar{N})^2 \\ &= \bar{r}_{uu}.\bar{r}_{vv} - (\bar{r}_{uu}.\bar{a})(\bar{r}_{vv}.\bar{a}) - (\bar{r}_{uu}.\bar{b})(\bar{r}_{vv}.\bar{b}) \\ &\quad - (\bar{r}_{uv}.\bar{r}_{uv} - (\bar{r}_{uv}.\bar{a})^2 - (\bar{r}_{uv}.\bar{b})^2) \end{aligned}$$

Terms such as $\bar{r}_{uu}.\bar{a}$, $\bar{r}_{uv}.\bar{b}$ contain E, F and G, $\bar{r}_{uu}.\bar{r}_u$, $\bar{r}_{uu}.\bar{r}_v$, $\bar{r}_{vv}.\bar{r}_v$, $\bar{r}_{vv}.\bar{r}_u$, $\bar{r}_{uv}.\bar{r}_u$ and $\bar{r}_{uv}.\bar{r}_v$.

However

$$\begin{aligned} \bar{r}_{uu}.\bar{r}_u &= \tfrac{1}{2}(\bar{r}_u.\bar{r}_u)_u = \tfrac{1}{2}E_u \\ \bar{r}_{uu}.\bar{r}_v &= (\bar{r}_u.\bar{r}_v)_u - \tfrac{1}{2}(\bar{r}_u.\bar{r}_u)_v = F_u - \tfrac{1}{2}E_v \\ \bar{r}_{uv}.\bar{r}_u &= \tfrac{1}{2}(\bar{r}_u.\bar{r}_u)_v = \tfrac{1}{2}E_v \end{aligned}$$

and so on.

This leaves $\bar{r}_{uu}.\bar{r}_{vv} - \bar{r}_{uv}.\bar{r}_{uv}$ in the expression for $eg - f^2$.

$$\begin{aligned} F_{uv} &= (\bar{r}_u.\bar{r}_v)_{uv} = (\bar{r}_{uu}.\bar{r}_v + \bar{r}_u.\bar{r}_{uv})_v \\ &= \bar{r}_{uuv}.\bar{r}_v + \bar{r}_{uu}.\bar{r}_{vv} + \bar{r}_{uv}.\bar{r}_{uv} + \bar{r}_u.\bar{r}_{uvv} \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(E_{vv} + G_{uu}) &= \tfrac{1}{2}((\bar{r}_u.\bar{r}_u)_{vv} + (\bar{r}_v.\bar{r}_v)_{uu}) \\ &= (\bar{r}_u.\bar{r}_{uv})_v + (\bar{r}_v.\bar{r}_{uv})_u \\ &= \bar{r}_{uv}.\bar{r}_{uv} + \bar{r}_u.\bar{r}_{uvv} + \bar{r}_{uv}.\bar{r}_{uv} + \bar{r}_v.\bar{r}_{uuv} \end{aligned}$$

Therefore $\qquad \bar{r}_{uu}.\bar{r}_{vv} - \bar{r}_{uv}.\bar{r}_{uv} = F_{uv} - \tfrac{1}{2}(E_{vv} + G_{uu})$

Thus the Gaussian curvature, K can be expressed as a function of E, G and F and their first and second derivatives with respect to u and v.

This is Gauss' Theorem and it means that as a surface is bent keeping E, G and F constant, the Gaussian curvature remains constant even though the individual principal normal curvatures change their values and directions. Bending of a surface can be imagined by making the surface out of thin sheet metal and then deforming it without changing lengths on the surface. Membrane shells require their boundaries to be restrained from moving to prevent such bending deformation.

The approach outlined above leads to Brioschi's expression for Gauss' theorem using determinants:

$$K = \frac{eg - f^2}{EG - F^2}$$

$$= \frac{1}{(EG - F^2)^2}\left[\begin{vmatrix} (-\tfrac{1}{2}E_{vv} + F_{uv} - \tfrac{1}{2}G_{uu}) & \tfrac{1}{2}E_u & (F_u - \tfrac{1}{2}E_v) \\ (F_v - \tfrac{1}{2}G_u) & E & F \\ \tfrac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \tfrac{1}{2}E_v & \tfrac{1}{2}G_u \\ \tfrac{1}{2}E_v & E & F \\ \tfrac{1}{2}G_u & F & G \end{vmatrix}\right] \qquad 5$$

The Weingarten Equations

$\bar{N}$ is aunit vector and hence $\bar{N}_u$ and $\bar{N}_v$ are perpendicular to $\bar{N}$. Thus $\bar{N}_u$ and $\bar{N}_v$ can be expressed as a combination of $\bar{r}_u$ and $\bar{r}_v$:

$$\bar{N}_u = \frac{fF - eG}{EG - F^2}\bar{r}_u + \frac{eF - fE}{EG - F^2}\bar{r}_v$$

$$\bar{N}_v = \frac{gF - fG}{EG - F^2}\bar{r}_u + \frac{fF - gE}{EG - F^2}\bar{r}_v$$

These equations are termed the Weingarten equations and are obtained by consideration of the fact the e, f and g are defined by equation 3f.

The Codazzi Equations

$$\frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} = (\bar{r}_{uu}.\bar{N})_v - (\bar{r}_{uv}.\bar{N})_u$$

$$= \bar{r}_{uuv}.\bar{N} + \bar{r}_{uu}.\bar{N}_v - \bar{r}_{uuv}.\bar{N} - \bar{r}_{uv}.\bar{N}_u$$

$$= \bar{r}_{uu}.\bar{N}_v - \bar{r}_{uv}.\bar{N}_u$$

$\bar{N}_u$ and $\bar{N}_v$ are given by the Weingarten equations and thus

$$\frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} = \frac{1}{2(EG - F^2)}\Big[e(GE_v - FG_u) + f(EG_u - GE_u + 2FF_u - 2FE_v)$$
$$+ g(EE_v + FE_u - 2EF_u)\Big] \qquad 6a$$

similarly

$$\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v} = \frac{1}{2(EG - F^2)}\Big[e(GG_u + FG_v - 2GF_v)$$
$$+ f(GE_v - EG_v + 2FF_v - 2FG_u) + g(EG_u - FE_v)\Big] \qquad 6b$$

These two equations are the Codazzi equations.

The Fundamental Theorem of surface theory

O. Bonnet proved that if E F, G and e, f, g are given as functions of u and v then the corresponding surface is uniquely defined except for its position in space. However the six quantities E, F, G and e, f, g cannot be given arbitarily as they must obey Gauss' theorem (equation 5) and the two Codazzi Equations (6a, 6b).

Geodesic coordinates

If any line is drawn on a surface then geodesics can be drawn crossing the line at right angles. Then a second set of lines can be drawn crossing each geodesic at right angles, the orthogonal trajectories of the geodesics (see fig. A4).

The geodesics can be chosen as the parametric lines u = constant and their orthogonal trajectories as the lines v = constant.

Since the coordinate lines are orthogonal, the fact that the geodesic curvature of the u = constant lines is zero is expressed by

$$\bar{r}_{vv}.\bar{r}_u = 0$$

Also $\quad \bar{r}_u.\bar{r}_v = F = 0$

hence $\quad \bar{r}_{uv}.\bar{r}_v + \bar{r}_u.\bar{r}_{vv} = 0$

Therefore $\bar{r}_{uv}.\bar{r}_v = \tfrac{1}{2}G_u = 0$

and G must be a function of v only.

We can always choose the parameter v such that along one particular geodesic (that is for one particular value of u) the value of G remains constant for all v. Hence since $G_u = 0$, G will remain constant throughout (fig. A5).

Since $F = G_u = G_v = 0$, the equation 5 expressing Gauss' theorem becomes much simplified:

$$K = \text{Gaussian curvature} = \frac{eg - f^2}{EG - F^2}$$

$$= \frac{1}{(EG)^2}\left(-\frac{1}{2}E\,G\,E_{vv} + \frac{1}{4}G\,E_v^{\,2}\right)$$

$$= \frac{E_v^{\,2} - 2EE_{vv}}{4E^2G} = \frac{-\sqrt{E}}{2EG}\left(\frac{E_v}{\sqrt{E}}\right)_v$$

$$\text{i.e. } K = -\frac{1}{G\sqrt{E}} \frac{\partial^2 \sqrt{E}}{\partial v^2} \qquad \text{——— } 7$$

Therefore if the parameter v is chosen such that $G = 1$ and Δ is the distance between two adjacent geodesics then

$$K = -\frac{1}{\Delta} \frac{\partial^2 \Delta}{\partial v^2} \qquad \text{——— } 8$$

The first of the Codazzi equations reduces to

$$\frac{\partial e}{\partial v} - \frac{\partial f}{\partial u} = \frac{1}{2EG} (eGE_v - fGE_u + gEE_v)$$

$$= \tfrac{1}{2}\left[\left(\frac{e}{E} + \frac{g}{G}\right) E_v - f \frac{E_u}{E}\right] \qquad \text{——— } 9a$$

and the second to

$$\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v} = \frac{1}{2EG} f G E_v$$

$$= \tfrac{1}{2} f \frac{E_v}{E} \qquad \text{——— } 9b$$

Equal mesh net

An equal mesh net can be laid over a surface by changing the angle α (fig. A6). If the net is chosen as the coordinate lines then

$$E = G = \text{constant}$$

Thus Gauss' theorem (equation 5) reduces to

$$K = \frac{eg-f^2}{EG - F^2} = \frac{1}{(EG - F^2)^2} \left[(EG - F^2) F_{uv} + FF_u F_v\right]$$

$$\text{but } F = \bar{r}_u . \bar{r}_v = \sqrt{(\bar{r}_u . \bar{r}_u)(\bar{r}_u . \bar{r}_v)} \cos \alpha$$

$$\text{i.e. } \cos \alpha = \frac{F}{\sqrt{EG}}$$

$$-\sin \alpha \frac{\partial \alpha}{\partial u} = \frac{F_u}{\sqrt{EG}}$$

$$-\sin \alpha \frac{\partial \alpha}{\partial v} = \frac{F_v}{\sqrt{EG}}$$

$$-\sin \alpha \frac{\partial^2 \alpha}{\partial u \partial v} - \cos \alpha \frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} = \frac{F_{uv}}{\sqrt{EG}}$$

Therefore

$$K = \frac{1}{(EG)^2 \operatorname{Sin}^4 \alpha} \left[(EG)^{3/2} \operatorname{Sin}^2 \alpha \left(-\operatorname{Sin} \alpha \frac{\partial^2 \alpha}{\partial u \partial v} - \operatorname{Cos} \alpha \frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right) + (EG)^{3/2} \operatorname{Cos} \alpha \operatorname{Sin}^2 \alpha \frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v}\right]$$

$$K = \frac{1}{\sqrt{EG} \operatorname{Sin} \alpha} \frac{\partial^2 \alpha}{\partial u \partial v} \qquad \text{——— } 10a$$

or if $E = G = 1$

$$K = -\frac{1}{\operatorname{Sin} \alpha} \frac{\partial^2 \alpha}{\partial u \partial v} \qquad \text{——— } 10b$$

APPENDIX B:

Equilibrium equations of membrane subject to internal pressure with principal stresses following the directions of geodesic coordinates on the surface.

Fig. B1 shows a part of a membrane surface. Q and T are the force/unit width in the direction of increasing u and v respectively. An internal pressure P is applied in the direction of the unit normal $\bar{N}$.

The equilibrium equations of the surface are found by considering equilibrium at the element ABCD bounded by the lines u, u $+\delta u$, v, v $+\delta v$. (δu, δv small)

Force crossing line CB

= unit vector in direction of force x force/unit width x length CB

$$= \frac{\bar{r}_u}{\sqrt{E}} \; Q \; (\sqrt{G}\,\delta v)$$

This expression must be evaluated at u $+\delta u$.

The force crossing line AD is given by a similar expression evaluated at u but is in the opposite direction. Therefore net force crossing CB and AD is

$$\frac{\partial}{\partial u}\left[\frac{\bar{r}_u}{\sqrt{E}} \; Q \; \sqrt{G}\right]\delta u\,\delta v$$

similarly the net force crossing AB and DC is

$$\frac{\partial}{\partial v}\left[\frac{\bar{r}_v}{\sqrt{G}} \; T \; \sqrt{E}\right]\delta u\,\delta v$$

The area of ABCD is equal to $\sqrt{EG}\,\delta u\,\delta v$ and therefore the force due to the pressure is $\sqrt{EG}\,P\,\bar{N}\,\delta u\,\delta v$

Therefore adding forces

$$\frac{\partial}{\partial u}\left[\frac{\bar{r}_u\,Q\,\sqrt{G}}{\sqrt{E}}\right] + \frac{\partial}{\partial v}\left[\frac{\bar{r}_v\,T\sqrt{E}}{\sqrt{G}}\right] + \sqrt{EG}\,P\,\bar{N} = 0 \qquad 11$$

This vector equation can be converted into three scalar equations:

Scalar multiplying by $\bar{r}_u$:

$$\sqrt{E}\;(Q\sqrt{G})_u = 0$$

(In arriving at this expression use was made of the facts that $\left(\frac{\bar{r}_u}{\sqrt{E}}\right)_u$ is perpendicular to $\bar{r}_u$ since $\frac{\bar{r}_u}{\sqrt{E}}$ is a unit vector,

that $\bar{r}_u.\bar{r}_v = 0$ and that $\bar{r}_u.\bar{r}_{vv} = 0$ since the u = constant lines are geodesics).

But $G_u = 0$ (since geodesic coordinates) therefore $Q_u = 0$ or $Q = V(v)$, a function of v only ——— 12

Scalar multiplying equation 11 by $\bar{r}_v$

$$\bar{r}_{uu}.\bar{r}_v \frac{Q\sqrt{G}}{\sqrt{E}} + \sqrt{G}\,(T\sqrt{E})_v = 0$$

but

$$(\bar{r}_u.\bar{r}_v)_u = \bar{r}_{uu}.\bar{r}_v + \bar{r}_u.\bar{r}_{uv} = 0$$

Therefore

$$-\tfrac{1}{2}\frac{E_v}{\sqrt{E}}\,Q + (T\sqrt{E})_v = 0$$

and

$$T\sqrt{E} = \tfrac{1}{2}\int \frac{E_v}{\sqrt{E}}\,Q\,dv + U(u)$$

$$= \sqrt{E}\,Q - \int \sqrt{E}\,\frac{dQ}{dv}\,dv + U(u) \qquad 13$$

(we can write $\frac{dQ}{dv}$ since $Q_u = 0$. U(u) is a function of integration).

Scalar multiplying equation 11 by $\bar{N}$

$$e\,Q\frac{\sqrt{G}}{\sqrt{E}} + g\,T\frac{\sqrt{E}}{\sqrt{G}} + \sqrt{EG}\;P = 0$$

or

$$Q\frac{e}{E} + T\frac{g}{G} + P = 0 \qquad 14$$

The three equilibrium equations 12, 13, 14 can be used for the form finding of air supported structures using the following procedure:

1. The stress in the surface is defined by the two functions V(v) and U(u).

2. The geometry of the surface is found such that it satisfies the equilibrium equation 14. The lines u = constant on this surface must be geodesics and the values of v along the geodesics be consistent with geodesic coordinates.

The simplest stress distribution in the surface is that with uniform stress (soap film) for which V(v) = constant, U(u) = 0.

References

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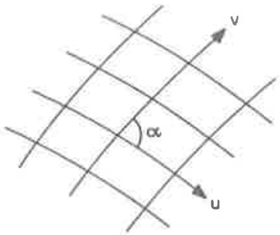


Fig 1.

Fig 2a.

Fig 2b.

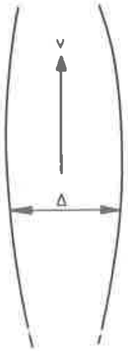
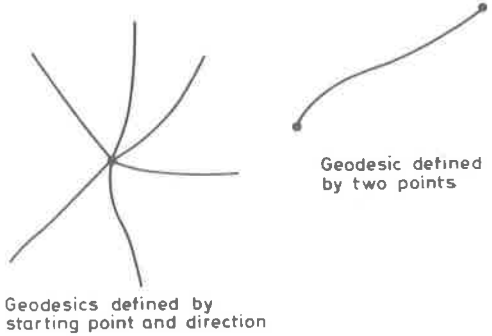


Fig 3.



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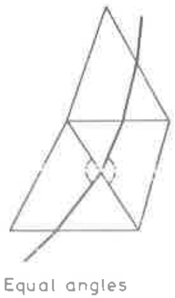

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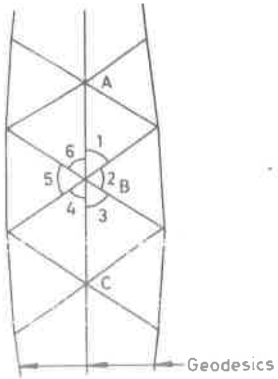

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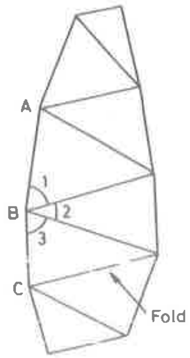

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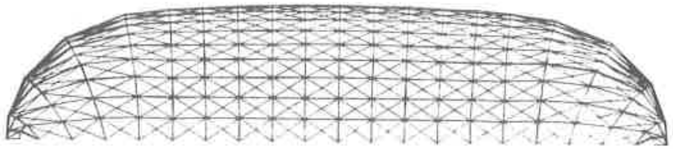

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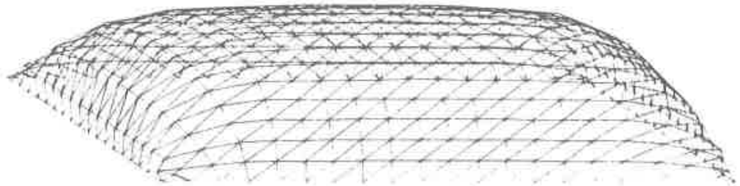

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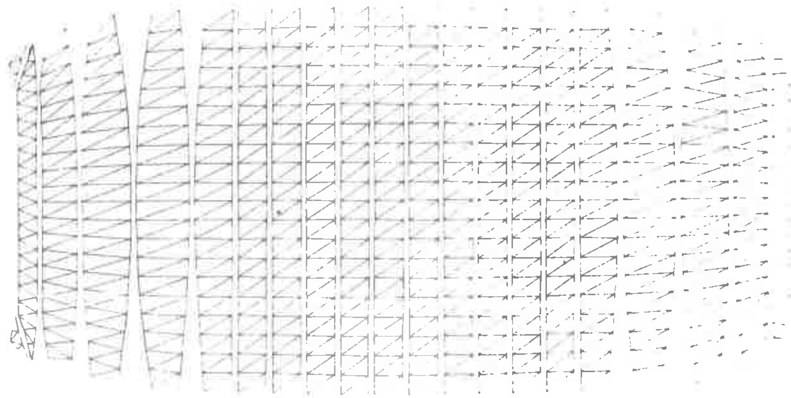


Figure 10

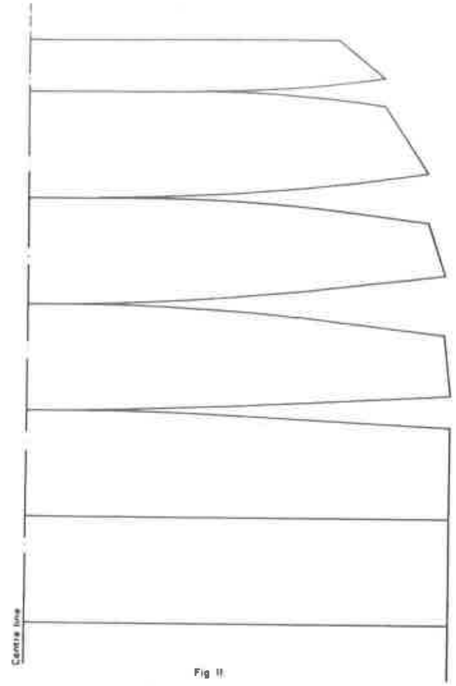


Fig 11

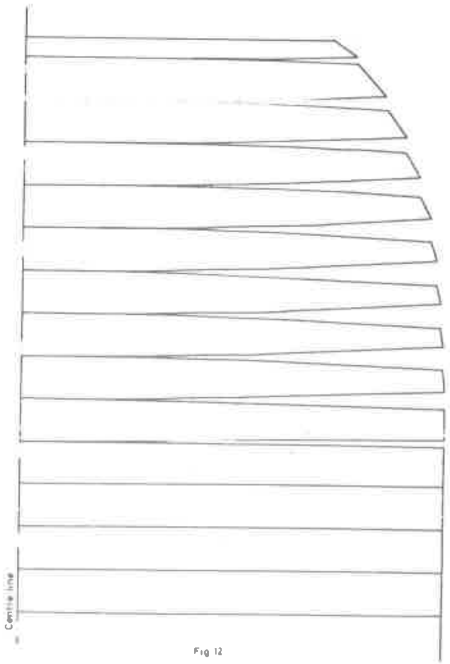


Fig 12

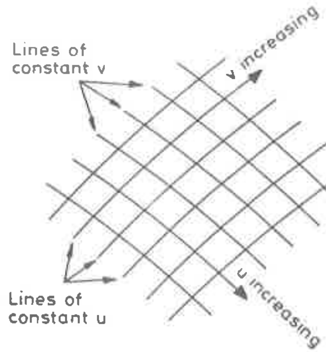


Fig A1

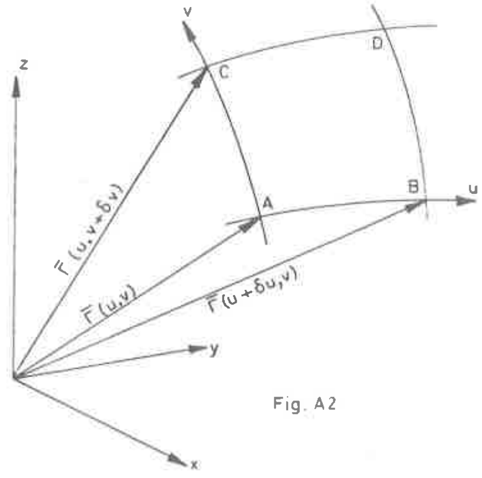
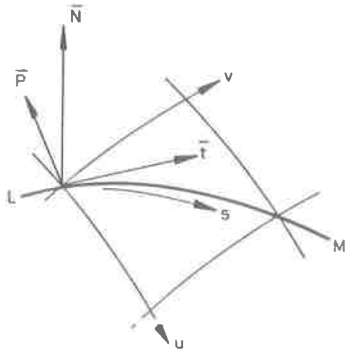


Fig. A2



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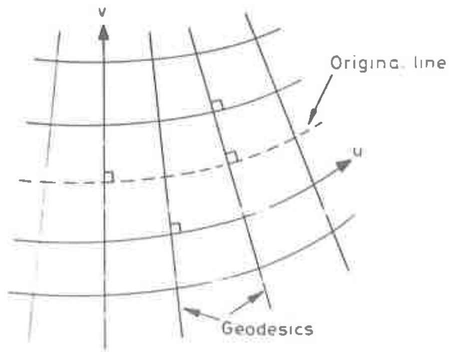
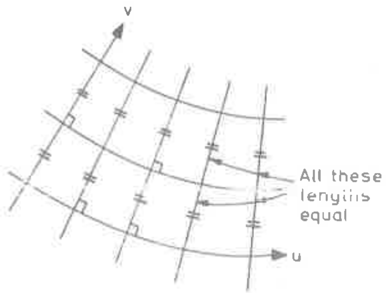


Fig A4



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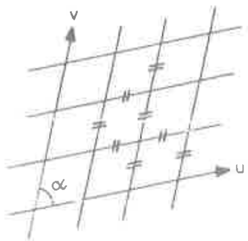

Fig A6

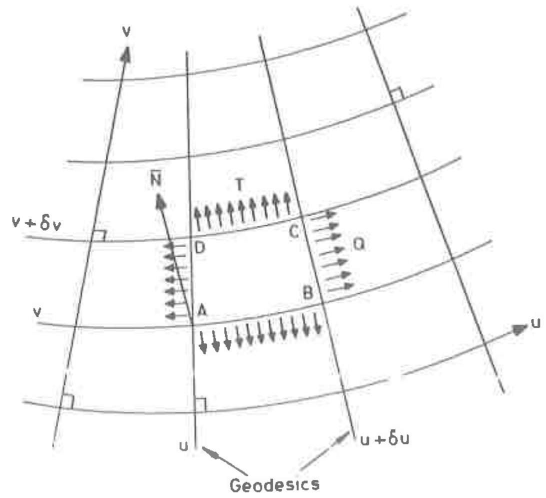

Fig. B1